SOME OPERATOR NORMALITIES ON BANACH SPACES

by

TWUM OWUSU-ANSAH

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "SOME OPERATOR NORMALITIES ON BANACH SPACES", submitted by TWUM OWUSU-ANSAH in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The introduction of semi-inner product spaces made possible two definitions of normal operators on a Banach space. In this paper we find the relation between the two kinds of normal operators and investigate some of their simple properties. We also use the semi-inner product in an attempt to extend the concept of hyponormal operators to Banach spaces.

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GLOSSARY

SYMBOL OR NAME	DEFINITION
$f(\alpha) = o(\alpha)$ as $\alpha \to o$,	$\frac{\lvert f(\alpha)\rvert}{\alpha} \to o$ as $\alpha \to o$
$\mathcal{C}(A)$, convex hull of a set A ,	The smallest convex set containing A .
$\rho(T)$, resolvent set of an operator T on a space X .	The set of all complex numbers λ such that $(\lambda I - T)^{-1}$ exists as a bounded linear operator with domain X .
$\sigma(T)$, the spectrum of T ,	The complement in the complex plane of the resolvent set of T .
$P\sigma(T)$, the point spectrum of T ,	The subset of $\sigma(T)$ for which $(\lambda I - T)^{-1}$ is not one-to-one.
$r(T)$, the spectral radius of T ,	$\sup \{ \lvert\lambda\rvert : \lambda \in \sigma(T) \}$.
$M_1 \oplus M_2$, the direct sum of the subspaces M_1, M_2 of a space X	The smallest subspace in X containing both M_1 and M_2 where $M_1 \cap M_2 = \{0\}$.
σ-Algebra of sets,	A family F of subsets of a set X such that $A_k, A, B \in F \Longrightarrow A \cap B \in F$, $X - A \in F$ and $\bigcup_{k=1}^{\infty} A_k \in F$.
Borel σ-Algebra of plane sets,	The smallest σ-Algebra of subsets of the complex plane containing all rectangles.

SYMBOL OF NAME	DEFINITION
T^* , the Hilbert space adjoint of an operator T ,	$(Tx,y) = (x,T^*y)$
Projection	A bounded linear operator on a Banach space X such that $P^2 = P$. In the case of a Hilbert space the condition $P^* = P$ is also required.
Algebra	A ring R which is a linear space and satisfies the condition $(\alpha x)y = \alpha(xy) = x(\alpha y)$ for $a \in C$, $x,y \in R$.
Division Algebra	An algebra such that every element $T \neq o$, has an inverse.
Boolean Algebra of projections	A distributive lattice R of projections containing two elements 0 and I such that (i) $0 \leq x \leq I$ for all $x \in R$ (ii) for any $x \in R$, there is some $x' \in R$ such that $x \vee x' = I$, $x \wedge x' = 0$.
Nilpolent operator T ,	An operator T for which there is some integer n such that $T^n = 0$.
Generalized quasi-nilpotent operator T	An operator T such that $\lim_{n \to \infty} \|T^n\|^{1/n} = 0$.
A positive operator T in a Hilbert space X , $T \geq 0$,	An operator T such that $(Tx,x) \geq 0$ for all $x \in X$.

CHAPTER I

PRELIMINARY RESULTS AND NOTATION

1.1 X will denote a complex Banach space and $B(X)$ will denote the associated Banach algebra of bounded linear operators on X.

1.2 Semi-inner product spaces

A semi-inner product on a linear space Y is a mapping $[\cdot,\cdot]$ from $Y \times Y$ into $\mathbb{C}$, the complex field such that

1.2.1 $[x+y,z] = [x,z] + [y,z]$

$[\lambda x,y] = \lambda[x,y]$, for all complex λ and all $x, y, z \in Y$.

1.2.2 $[x,x] > 0$ if $x \neq 0$, $x \in Y$

1.2.3 $|[x,y]|^2 \leq [x,x][y,y]$, x and $y \in Y$.

A semi-inner product space (S.I.P.S) is a linear space on which a semi-inner product is defined. These definitions were introduced by G. Lumer [12] who proved the following results

1.2.4 Every semi-sinner product space is normed, the norm of a point x being $[x,x]^{1/2}$

1.2.5 Every normed linear space can be made into a semi-inner product space in such a way that the norm induced by the semi-inner product on the normed space coincides with the norm on the space. [In general this can be done in infinitely many ways. There is a unique semi-inner product on a normed space if and only if the unit sphere of the space is smooth. In particular the unique semi-inner product on a Hilbert space is the inner product.]

1.3 <u>The numerical range of an operator $T \in B(X)$</u> .

The set of complex numbers

$$W(T) = \{[Tx,x] : [x,x] = 1 \ , \ x \in X\} \ .$$

where $[\cdot,\cdot]$ denotes one of the semi-inner products that can be introduced on the space X in the manner of 1.2.5, is called a numerical range for the operator T . It follows that there are many numerical ranges of an operator in general, depending on which semi-inner product is introduced on X . This definition is due to G. Lumer [12] who also proved the following results:

1.3.1 The convex hull of a numerical range for an operator on a normed linear space is independent of the semi-inner product used, as long as the semi-inner product yields the norm.

1.3.2 In particular, if the numerical range of an operator T is real for one semi-inner product introduced on X in the

manner of 1.2.5, then it is real for all other semi-inner products.

1.4 Hermitian Operators

An operator $T \in B(X)$ is said to be hermitian if and only if its numerical range with respect to some semi-inner product introduced on X in the manner of 1.2.5 is real. By virtue of 1.3.2, it is clear that this concept of hermiticity does not depend on the semi-inner product used. The following result is due to G. Lumer [12].

1.4.1 An operator $T \in B(X)$ is hermitian if and only if

$$\|I+i\alpha T\| = 1 + o(\alpha) \quad \text{as} \quad \alpha \to o ,$$

where α is real and I is the identity operator in $B(X)$.

This result establishes the equivalence of G. Lumer's definition of hermiticity and that given by I. Vidav [19].

1.5 Reducibility of an operator $T \in B(X)$.

An operator $T \in B(X)$ is said to be completely reduced by a pair of sub-spaces M_1, M_2 of X if and only if

1.5.1 $\quad X = M_1 \oplus M_2$

1.5.2 $$TM_1 \subset M_1 \ , \quad TM_2 \subset M_2 \ .$$

(See [18, page 268].)

1.6 Decomposition Theorem.

Let $T \in B(X)$. An admissible domain for T is any bounded open set D in the complex plane whose boundary ∂D consists of a finite number of closed rectifiable curves lying in the resolvent set $\rho(T)$ of T and oriented according to the usual convention in the complex plane. An isolated part σ of $\sigma(T)$, the spectrum of T , is any subset of $\sigma(T)$ which is at a positive distance from its complementary part $\sigma' = \sigma(T) - \sigma$. If σ is an isolated part of $\sigma(T)$, then there exists an admissible domain D for T such that $\sigma = \sigma(T) \cap D$. In this case we have the following decomposition theorem [16, p. 420-421].

There exist subspaces M_σ , $M_{\sigma'}$ of X such that

1.6.1 $$X = M_\sigma \oplus M_{\sigma'}$$

1.6.2 $$TM_\sigma \subset M_\sigma \ , \ TM_{\sigma'} \subset M_{\sigma'}$$

1.6.3 $$\text{Spectrum of } \ T|_{M_\sigma} = \sigma$$

1.6.4 $$\text{Spectrum of } \ T|_{M_{\sigma'}} = \sigma(T) - \sigma = \sigma'$$

1.7 Spectro-normaloid operators

An operator $T \in B(X)$ such that $\|T\| = \max \{|\lambda| : \lambda \in \sigma(T)\}$, where $\sigma(T)$ denotes the spectrum of T, will be called spectro-normaloid. In Hilbert space an operator T is spectro-normaloid if and only if it is normaloid [10, p. 416], i.e. $\|T\| = \max \{|\lambda| : \lambda \in W(T)\}$.

1.8 The property SN

A set of operators $S \subset B(X)$ such that every member is spectro-normaloid will be said to have property SN.

1.9 Spectral Operators of scalar type

Let $\mathcal{A}$ be the σ-algebra of Borel subsets of the complex plane $\mathbb{C}$. Let X be an arbitrary Banach space over the complex field. The function $p(.) : \mathcal{A} \to B(X)$ is called a spectral measure in X iff the following hold for all $\Delta_1, \Delta_2, \Delta_3 \in \mathcal{A}$:

1.9.1 $p(\Delta_1 \cap \Delta_2) = p(\Delta_1)p(\Delta_2)$

1.9.2 $p(\Delta_1 \cup \Delta_2) = p(\Delta_1) + p(\Delta_2) - p(\Delta_1)p(\Delta_2)$

1.9.3 $p(\mathbb{C} - \Delta) = I - p(\Delta)$, where I is the identity operator in $B(X)$.

1.9.4 There exists a constant M such that $\|p(\Delta)\| \leq M$ for all $\Delta \in \mathcal{A}$.

It follows from 1.9.1 that $p(\Delta) = \{p(\Delta)\}^2$ for all $\Delta \in \mathcal{A}$ and hence that $p(\cdot)$ is projection-valued. Thus the range of $p(\cdot)$ is a bounded Boolean algebra of projections.

A subset Γ of X', the conjugate space of X, is said to be total if $f(x) = 0$ for every $f \in \Gamma$ implies $x = 0$. If Γ is a total subset of X', then the spectral measure $p(\cdot)$ is said to be Γ-completely additive if

1.9.5 For any pairwise disjoint sequence $\{\Delta_k\}_{k=1}^{\infty}$ of $\mathcal{A}$ we have for all $x \in X$ and all $f \in \Gamma$,

$$f\{p(\bigcup_{k=1}^{\infty} \Delta_k)x\} = \sum_{k=1}^{\infty} f\{p(\Delta_k)x\}$$

An operator $T \in B(X)$ is called a spectral operator if there exists a Γ-completely additive spectral measure $p(\cdot)$ in X such that the following conditions hold:

1.9.6 T commutes with $p(\Delta)$ for all $\Delta \in \mathcal{A}$.

1.9.7 Let $X_\Delta = p(\Delta)X$ and $T_\Delta = p(\Delta)T$. If $u \in X_\Delta$, then $u = p(\Delta)x$ for some $x \in X$. Hence

$$\begin{aligned} T_\Delta u &= p(\Delta)Tu \\ &= p(\Delta)Tp(\Delta)x \\ &= p(\Delta)^2 Tx , \end{aligned}$$

assuming 1.9.6 holds. That is $T_\Delta u = p(\Delta)Tx$ since

$$\{p(\Delta)\}^2 = p(\Delta) .$$

Thus $T_\Delta u = p(\Delta)(Tx) \in X_\Delta$. Hence $T_\Delta u \in X_\Delta$ and so T_Δ maps X_Δ into itself. That is $T_\Delta \in B(X_\Delta)$. The condition to be satisfied is that the spectrum of T_Δ considered as an element of $B(X_\Delta)$ must be contained in the closure of Δ .

When T is a spectral operator the spectral measure $p(\cdot)$ is called a resolution of the identity for T . For such an operator T , N. Dunford [5, pages 329-330] has proved the following results:

1.9.8 $p(\cdot)$ is unique

1.9.9 Any operator which commutes with T , commutes with $p(\Delta)$ for all $\Delta \in \mathscr{A}$.

Let T be a spectral operator with resolution of the identity $p(\cdot)$. Let $\sigma(T)$, the spectrum of T , which is compact, be partitioned into disjoint non-void Borel sets $\Delta_1, \Delta_2, \ldots, \Delta_n$ whose diameters are each less than δ , δ being an arbitrary positive number. Let us define $\|\pi\|$, the norm of such a partition $\pi = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ as $\max\limits_{1 \leq i \leq n} \{\operatorname{diam} \Delta_i\}$. For a scalar-valued or operator-valued function f defined on $\sigma(T)$ if

$$\lim_{\|\pi\| \to o} \sum_{i=1}^{n} f(\lambda_i)p(\Delta_i)$$

exists independently of the choice of π and $\lambda_i \in \Delta_i$ $\{i = 1,2,\ldots,n\}$, we say that f is integrable and define

$$\int_{\sigma(T)} f(\lambda)p(d\lambda) ,$$

the integral of f , as the limit

$$\lim_{\|\pi\| \to o} \sum_{i=1}^{n} f(\lambda_i)p(\Delta_i)$$

which is taken in the uniform operator topology. Thus

1.9.10 $$\int_{\sigma(T)} f(\lambda)p(d\lambda) = \lim_{\|\pi\| \to o} \sum_{i=1}^{n} f(\lambda_i)p(\Delta_i)$$

A spectral operator T such that

$$T = \int_{\sigma(T)} \lambda p(d\lambda)$$

is said to be of scalar type.

1.10 Gelfand Theory of Commutative Banach Algebras

A Banach algebra Y is an algebra which is also Banach space such that $\|x\,y\| \leq \|x\|\,\|y\|$ for all $x,y \in Y$. Let Y denote a complex commutative Banach algebra with identity e of norm 1 . A non-empty subset I of Y is called an ideal if

1.10.1 I is a subspace of Y

1.10.2 $\quad x \in I\ ,\ \ y \in Y \Longrightarrow xy \in I$

An ideal I is proper if $I \neq Y$. A proper ideal M is said to be maximal if the only ideal that contains M properly is all of Y. A maximal ideal is closed. Since Y is a complex commutative Banach algebra with identity e, it follows that the quotient algebra Y/M is also a complex commutative Banach algebra with identity $e + M$, M being any maximal ideal.

Now Y is also a commutative ring with identity and since M is a maximal ideal in Y, it follows by the Gelfand-Mazur theorem that since Y/M is a complex Banach algebra with identity as well as a division algebra, it is isomorphic to the complex field. It is also clear from the proof of this result [1, p. 42] that given y and M, there exists a complex number $y(M)$ which corresponds to $y + M$ in the isomorphism, such that $y + M = y(M)\ (e + M)$. Thus using the natural homomorphism $Y \rightarrow Y/M$ we may map Y onto the complex field as follows

$$Y \rightarrow Y/M \rightarrow \text{complex field}$$

$$\text{where} \quad y \rightarrow y + M \rightarrow y(M) \quad .$$

Suppose we now fix y and let M vary over the set of all maximal ideals $\mathcal{M}$ in Y. Then y determines a unique function $\hat{y}$ defined over $\mathcal{M}$ such that $\hat{y}(M) = y(M)$. Thus if we let $C(\mathcal{M})$ denote the set of all these complex-valued functions $\hat{y}$,

we obtain a homomorphism from Y into the algebra of complex-valued functions $C(\mathcal{M})$.

This homomorphism has the following properties:

1.10.3 $\quad \widehat{x_1 + x_2} = \hat{x}_1 + \hat{x}_2$

1.10.4 $\quad \widehat{\alpha x} = \alpha\,\hat{x}$

1.10.5 $\quad \widehat{x_1 x_2} = \hat{x}_1\,\hat{x}_2$

1.10.6 $\quad \hat{e} = 1$

We also have the properties

1.10.7 $\quad \sigma(y) = \{y(M) : M \in \mathcal{M}\}$

1.10.8 $\quad r(y) = \sup\,\{|y(M)| : M \in \mathcal{M}\}$

The set $\mathcal{M}$ can be topologised by giving it the weakest topology with respect to which all the functions $\{\hat{y}\}_{y \in Y}$ are continuous. This is the Gelfand topology on $\mathcal{M}$. With respect to this topology we have the following results:

1.10.9 $\quad \mathcal{M}$ is a compact Hausdorff space

1.10.10 $\quad$ All the functions $\{\hat{y}\}_{y \in Y}$ are continuous on $\mathcal{M}$.

1.11 Some results due to E. Berkson

1.11.1 If A is a commutative Banach algebra and if every $a \in A$ can be written in the form $a = s + it$, where s and t are hermitian operators belonging to A, then for all $a \in A$, we have $\|a\| = r(a)$, the spectral radius of a. [3, p. 5]

1.11.2 Suppose X is reflexive. If T is a spectral operator of scalar type then there exist operators $u, v \in B(X)$ such that (i) $T = u + iv$ (ii) $uv = vu$, (iii) relative to some norm on X equivalent to the given norm, $u^m v^n$ are all hermitian for $m, n = 0, 1, 2, \ldots$.

[2, p. 371]

1.11.3 If X is reflexive and $T = u + iv$ where $u, v \in B(X)$, $uv = vu$ and relative to some norm on X equivalent to the given one, we have $u^m v^n$ hermitian for $m, n = o$, $1, 2, \ldots,$ then T is spectral of scalar type.

[2, p. 371]

1.12 A result due to S. R. Foguel

If $T \in B(X)$ is spectral of scalar type, then there exist operators $R, S \in B(X)$ such that

(i) $T = RS = SR$

(ii) $\sigma(R)$ is a set of non-negative real numbers and $\sigma(S)$ is a

subset of the unit circle.

[Theorem 2, 7, p. 61]

1.13 Normal Operators on a Hilbert Space

If T is a normal operator on a Hilbert space, the following hold (see [16] and [18]):

1.13.1 $T = \int_{\sigma(T)} \lambda \; p(d\lambda)$ where $p(\cdot)$ is the resolution of the identity for T.

1.13.2 λT and $\lambda + T$ are normal for all complex λ

1.13.3 $\|T\| = r(T)$

1.13.4 T^n is normal for all non-negative integral n

1.13.5 There exist operators R, S where R is unitary and S is positive such that

$$T = RS = SR$$

1.13.6 T^*, the adjoint of T is normal and conversely.

1.13.7 Any operator which commutes with T, commutes with $p(\Delta)$ for all Δ.

1.13.8 If S is an operator such that $ST = TS$, then $S + T$ and TS are both normal.

1.13.9 If a subspace M of X reduces T, then $T|_M$ is normal.

1.13.10 The limit in the uniform operator topology of a sequence of normal operators is normal.

1.14 Hyponormal operators on a Hilbert space.

An operator T on a Hilbert space H is said to be hyponormal if $\|T^*x\| \leq \|Tx\|$ for all $x \in H$. The following results hold:

1.14.1 T is hyponormal if and only if

$$T^*T - TT^* \geq 0 \quad .$$

Proof:

Suppose T is hyponormal. Then for all $x \in H$,

$$\begin{aligned}(T^*Tx,x) = (Tx,Tx) = \|Tx\|^2 &\geq \|T^*x\|^2 \\ &= (T^*x,T^*x) \\ &= (TT^*x,x)\end{aligned}$$

i.e. $T^*T - TT^* \geq 0$.

Now suppose $T^*T - TT^* \geq 0$. Then for all $x \in H$,

$$\begin{aligned}\|T^*x\|^2 = (T^*x,T^*x) = (TT^*x,x) &\leq (T^*Tx,x) \\ &= (Tx,Tx) \\ &= \|Tx\|^2\end{aligned}$$

i.e. T is hyponormal.

1.14.2 If T is hyponormal then so are λT and $T + \lambda$ for all complex λ .

Proof:

Since T is hyponormal, we have

$$([T^*T - TT^*]x,x) \geq 0 \quad \text{for all} \quad x \in X \quad .$$

Now

$$([(\lambda T)^*(\lambda T) - (\lambda T)(\lambda T)^*]x,x)$$

$$= ([\bar{\lambda} T^*\lambda T - \lambda T\bar{\lambda} T^*]x,x)$$

$$= |\lambda|^2 ([T^*T - TT^*]x,x) \geq 0 \quad .$$

Hence λT is hyponormal

$$([T+\lambda)^*(T+\lambda) - (T+\lambda)(T+\lambda)^*]x,x)$$

$$= ([(T^*+\bar{\lambda})(T+\lambda) - (T+\lambda)(T^*+\bar{\lambda})]x,x)$$

$$= ([T^*T - TT^*]x,x) \geq 0 \quad .$$

Hence $T + \lambda$ is hyponormal.

1.14.3 If T is hyponormal, then $\|T\| = r(T)$, the spectral radius of T ,

[Theorem 1, p. 1453, 17]

1.14.4 If T is hyponormal and $M \subset X$ is invariant under T , then $T|_M$ is also hyponormal.

Proof:

$$\|T^* x\| \leq \|Tx\| \quad \text{for all} \quad x \in H \quad .$$

In particular,

$$\|T^* x\| \leq \|Tx\| \quad \text{for all} \quad x \in M \quad .$$

Hence we have

1.14.4.1 $$\|T^* |_M x\| \leq \|T |_M x\| \quad .$$

Now $M \subset X$ is closed subspace and so is a Hilbert space. It follows then that

$$(T|_M)^* : M^* \to M^*$$

and so

$$(T|_M)^* : M \to M \quad .$$

But

$$T^* : X^* \to X^* \quad ,$$

that is

$$T^* : X \to X \quad .$$

Thus

$$T^* |_M : M \to M \quad .$$

Let $x \in M$ be arbitrary. Then for all $y \in M$ we have

$$\begin{aligned} ([T|_M]^* x, y) &= (x, [T|_M] y) \\ &= (x, Ty) \\ &= (T^* x, y) \\ &= (T^* |_M x, y) \quad . \end{aligned}$$

In particular, putting $y = (T|_M)^* x \in M$ we have

$$([T|_M]^* x, [T|_M]^* x) = (T^*|_M x, [T|_M]^* x) \quad .$$

Thus

$$\begin{aligned} \|(T|_M)^* x\|^2 &= |(T^*|_M x, [T|_M]^* x)| \\ &\leq \|T^*|_M x\| \cdot \| (T|_M)^* x\| \end{aligned}$$

by Schwarz's inequality

$$\leq \|(T|_M)x\| \cdot \|(T|_M)^* x\| \quad ,$$

by 1.14.4.1.

Hence

$$\|(T|_M)^* x\| \leq \|(T|_M)x\|$$

for all $x \in M$ and so $T|_M$ is hyponormal.

CHAPTER II

PSEUDONORMAL AND NORMAL OPERATORS

2.1 Pseudonormal Operators

An operator $T \in B(X)$ will be called pseudonormal if and only if it can be written in the form $T = u + iv$ where u and v are hermitian operators in $B(X)$ such that $uv = vu$. Lumer [13] called these operators normal and showed that u and v are unique. We shall call u and v the components of the pseudonormal operator T.

2.2 Normal Operators

An operator $T \in B(X)$ will be called normal if and only if it can be written in the form

$$T = u + iv \,, \quad \text{where} \quad uv = vu \,,$$

u and $v \in B(X)$ and $u^m \; v^n$ are all hermitian operators for $m, n = 0,1,2,\ldots$. This definition is due to E. Berkson [3]. Clearly normal operators are pseudonormal.

2.3 Theorem: Let $u, v \in B(X)$ be two commuting operators. Let $B_o(X)$ be the closed subalgebra of $B(X)$ generated by I, u and v. Then u and v hermitian implies $u^m v^n$ hermitian for all $m, n = 0, 1, 2, \ldots$ if and only if $B_o(X)$ has property SN.

Proof:

(i) Sufficiency

$B_o(X)$ is a commutative Banach algebra with identity. Hence by the Gelfand representation theory (1.10) we have a homomorphism from $B_o(X)$ into $C(\mathcal{M})$, the space of continuous functions over the maximal ideal space $\mathcal{M}$. Since u and v are hermitian operators, $\sigma(u)$ and $\sigma(v)$ are real [9, p. 548]* . Thus by 1.10.7 we have

2.3.1 $U(M)$ and $V(M)$ are real for all $M \in \mathcal{M}$

Let t be real. Since $I + it\, u^m v^n \in B_o(X)$ and $B_o(X)$ has property SN, it follows that $I + it\, u^m v^n$ is spectro-normaloid and so by 1.10.8 we have

$$\|I + it\, u^m v^n\| = r\{I + it\, u^m v^n\}$$

$$= \sup_{M \in \mathcal{M}} |(I + it\, u^m v^n)(M)| \quad .$$

---

* This also follows from a more general result of J. P. Williams. See [20],

Thus

$$2.3.2 \qquad \|I + it\, u^m v^n\| = \sup_{M \in \mathcal{M}} |(I + it\, u^m v^n)(M)|$$

Since $(I + it\, u^m v^n)(M)$ is a complex-valued continuous function over the compact Hausdorff space $\mathcal{M}$, by 1.10.9 and 1.10.10, it follows that $|(I + it\, u^m v^n)(M)|$ is a real-valued continuous function over $\mathcal{M}$. Hence there exists $M_o \in \mathcal{M}$ such that

$$\sup_{M \in \mathcal{M}} |(I + it\, u^m v^n)(M)| = |(I + it\, u^m v^n)(M_o)|$$

Hence by 1.10.3 to 1.10.6 we have

$$\begin{aligned}\sup_{M \in \mathcal{M}} |(I + it\, u^m v^n)(M)| &= |(I + it\, u^m v^n)(M_o)| \\ &= |I(M_o) + it\{u(M_o)\}^m\{v(M_o)\}^n| \\ &= |1 + it\,\lambda| \quad ,\end{aligned}$$

where $\lambda = \{u(M_o)\}^m\{v(M_o)\}^n$ is real by 2.3.1. Hence by 2.3.2 we have

$$\begin{aligned}\|I + it\, u^m v^n\| &= |1 + it\,\lambda| \\ &= \sqrt{1 + t^2\lambda^2} \quad , \text{ since } \lambda t \text{ is real} \\ &= 1 + \tfrac{1}{2}\, t^2\lambda^2 + \dots \\ &= 1 + o(t) \ , \quad \text{as} \quad t \to o \quad .\end{aligned}$$

Thus by the equivalence of Lumer's and Vidav's hermiticity [1.4.1] it follows that $u^m v^n$ are hermitian operators for all $m,n = 0,1,2,\ldots$.

(ii) Necessity

We shall use the following lemma which was essentially established by E. Berkson [2]. A variant of his proof will be given here for the sake of completeness:

2.3.3 Lemma

If B is a subset of $B(X)$ such that every $b \in B$ can be written in the form $b = s + it$ where s and t are hermitian operators in B, then every $c \in \overline{B}$, the closure of B, can be written in the form $c = s_o + it_o$ where s_o and t_o are hermitian operators in $\overline{B}$.

By hypothesis, we have $uv = vu$ and $u^m v^n$ hermitian for all $m,n = 0,1,2,\ldots$.

Let B denote the class of polynomials in u and v with complex coefficients. Then B is a commutative subalgebra of $B(X)$. $\overline{B}$, being the closed subalgebra of $B(X)$ generated by I, u and v is also commutative. Thus $\overline{B} = B_o(X)$ is a commutative Banach algebra. Since every $P \in B$ is a polynomial in u and v with complex coefficients, each $P \in B$ can be written $P = R + iJ$, where R and J are polynomials in u and v with real coefficients. Since

$u^n v^m$ are all hermitian for $m,n = 0,1,2,\ldots,$ R and J are also hermitian. Furthermore R and J belong to B .

Hence by lemma 2.3.3, every element $C \in \overline{B} = B_o(X)$ can be written in the form $C = R_o + i\,J_o$ where R_o and J_o are hermitian and belong to $B_o(X)$. Thus $B_o(X)$ is a commutative Banach algebra such that every $C \in B_o(X)$ can be written in the form $C = R_o + i\,J_o$ where R_o and J_o are hermitian operators in $B_o(X)$. It follows by 1.11.1 that for all $T \in B_o(X)$ we have

$$\|T\| = r(T) \quad .$$

Thus $B_o(X)$ has property SN .

Now to prove lemma 2.3.3, let $C \in \overline{B}$. Then there exists a sequence $\{c_n\}_{n=1}^{\infty} \in B$ such that $c_n \to c$ as $n \to \infty$.

Since $c_n \in B$, it follows that $c_n = s_n + i\,t_n$ where s_n and t_n are hermitian operators belonging to B .

Let $\|x\| = 1 = [x,x]^{1/2}$

Then

$$\begin{aligned} &|[(s_n-s_m)x,x] + i[t_n-t_m)x,x]| \\ = \; &|[\{(s_n+it_n) - (s_m + it_m)\}x,x]| \\ = \; &|[(c_n-c_m)x,x]| \quad . \end{aligned}$$

Hence by 1.2.3, we have

2.3.5 $$|[(s_n-s_m)x,x] + i[(t_n-t_m)x,x]| \le \|c_n-c_m\| .$$

Since s_n, s_m, t_n, t_m are all hermitian, so are s_n-s_m and t_n-t_m. Hence $[(s_n-s_m)x,x]$, $[(t_n-t_m)x,x]$ are both real. Thus

$$|[(s_n-s_m)x,x] + i[(t_n-t_m)x,x]| = \sqrt{[(s_n-s_m)x,x]^2 + [(t_n-t_m)x,x]^2} .$$

It follows by 2.3.5 that we have

2.3.6 $$|[(s_n-s_m)x,x]| \le \|c_n-c_m\|$$
$$|[(t_n-t_m)x,x]| \le \|c_n-c_m\|$$

But for all operators $T \in B(X)$ we have [2, p. 366]

$$\sup_{\|x\|=1} |[Tx,x]| \le \|T\| \le 4 \sup_{\|x\|=1} |[Tx,x]| .$$

Thus by 2.3.6 we have

2.3.7 $$\|s_n-s_m\| \le 4\|c_n-c_m\|$$
$$\|t_n-t_m\| \le 4\|c_n-c_m\| .$$

Since $\{c_n\}_{n=1}^{\infty}$ is convergent, it follows by 2.3.7 that both sequences $\{s_n\}_{n=1}^{\infty}$, $\{t_n\}_{n=1}^{\infty}$ are convergent.

Let $\mathop{Lt}\limits_{n \to \infty} t_n = t_o$, $\mathop{Lt}\limits_{n \to \infty} s_n = s_o$.

Then $c = \mathop{Lt}\limits_{n \to \infty} c_n = s_o + it_o$.

Let ϵ be an arbitrary positive real number. Then there exists $N = N(\epsilon)$, a positive integer, such that for all $m > N$ we have

2.3.8 $$\|s_o - s_m\| < \epsilon, \quad \|t_o - t_m\| < \epsilon \quad .$$

Hence
$$\begin{aligned} \|I + its_o\| &= \|I + its_m - its_m + its_o\| \\ &\leq \|I + its_m\| + |t| \, \|s_o - s_m\| \\ &\leq 1 + o(t) + |t| \, \epsilon \quad \text{as} \quad t \to 0 \, . \end{aligned}$$

Since ϵ is arbitrary and the left hand side is independent of it, it follows that

$$\|I + its_o\| \leq 1 + o(t) \quad \text{as} \quad t \to 0 \quad .$$

By a remark in [19, p. 122] this implies

$$\|I + its_o\| = 1 + o(t) \quad \text{as} \quad t \to 0 \quad .$$

Hence s_o is a hermitian operator. Similarly t_o is a hermitian operator.

Since t_n and s_n belong to B, s_o and t_o belong to $\bar{B}$. Thus $c = s_o + it_o$ where s_o and t_o are hermitian operators belonging to $\bar{B}$.

2.4 Remarks

2.4.1 If X is a Hilbert space, then since the product of any two commuting hermitian operators is hermitian, we have by 2.3 that the closed subalgebra of $B(X)$ generated by any two commuting hermitian operators and the identity operator always has property SN. From a result of C. A. McCarthy's [14], G. Lumer [13] has deduced the existence of a hermitian operator $L \in B(X)$, where X is reflexive but non-Hilbert, whose powers are not all hermitian. In this space X, the closed subalgebra of $B(X)$ generated by two commuting hermitian operators and the identity operator need not have property SN.

2.4.2 Using the operator L in 2.4.1 let us form the family $L + \lambda I$, where λ is real and I is the identity operator. Each member of this family is pseudonormal but not normal.

2.4.3 It is clear from 2.3 that a pseudonormal operator which is such that the closed subalgebra of $B(X)$ generated by its components and the identity operator has property

SN , is normal. Thus on a Hilbert space pseudonormal and normal operators coincide.

2.5 Some properties of normal and pseudonormal operators

2.5.1 Let X be reflexive. If $T \in B(X)$ is normal, then T is spectral of scalar type and so has the spectral representation

$$T = \int_{\sigma(T)} \lambda P(d\lambda) \quad ,$$

where $P(.)$ is the resolution of the identity for T. If T is spectral of scalar type, then there exists a norm on X equivalent to the given norm relative to which T is normal.

Proof:

Since T is normal, we have $T = u + iv$ where $uv = vu$ and $u^m v^n$ are hermitian operators for all $m,n = 0,1,2,\ldots$. Hence by 1.11.3, T is spectral of type and so

$$T = \int_{\sigma(T)} \lambda P(d\lambda) \quad .$$

Now suppose T is spectral of scalar type. Then by 1.11.2 there exist operators $u,v \in B(X)$ such that $T = u + iv$, $uv = vu$ and relative

to some norm on X equivalent to the given norm, $u^m v^n$ are hermitian operators for $m,n = 0,1,2,\ldots$. Relative to this norm, T is normal.

2.5.2 If $T \in B(X)$ is normal, then so are λT and $T + \lambda$ for all complex λ .

<u>Proof</u>:

Let $T = u + iv$, where $uv = vu$ and $u^m v^n$ are hermitian operators for $m,n = 0,1,2,\ldots$.

Let $\lambda = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real. Then

$$\lambda T = (\lambda_1 + i\lambda_2)(u + iv)$$

$$= (\lambda_1 u - \lambda_2 v) + i(\lambda_2 u + \lambda_1 v) \quad .$$

Since λ_1, λ_2 are real, $(\lambda_1 u - \lambda_2 v)^M(\lambda_2 u + \lambda_1 v)^N$ is a polynomial in u and v with real coefficients for $M,N = 0,1,2,\ldots$. Since every term $u^m v^n$ is hermitian it follows that $(\lambda_1 u-\lambda_2 v)^M(\lambda_2 u+\lambda_1 v)^N$ is also hermitian.

Now $(\lambda_1 u-\lambda_2 v)(\lambda_2 u+\lambda_1 v)$

$$= \lambda_1\lambda_2 u^2 + \lambda_1^2 vu - \lambda_2^2 uv - \lambda_2\lambda_1 v^2 \text{ , since } uv = vu$$

$$= (\lambda_2 u + \lambda_1 v)(\lambda_1 u - \lambda_2 v) \quad .$$

Hence λT is normal.

$$T + \lambda = (u + iv) + (\lambda_1 + i\lambda_2)$$

$$= (u + \lambda_1) + i(v + \lambda_2) \quad .$$

For $M,N = 0,1,2,\ldots$, $(u + \lambda_1)^M(v + \lambda_2)^N$ is a polynomial in u and v with real coefficients and since every term $u^m v^n$ is hermitian for $m,n = 0,1,2,\ldots$, it follows that $(u + \lambda_1)^M(v + \lambda_2)^N$ is also hermitian for all $M,N = 0,1,2,\ldots$.

Now $\quad (u + \lambda_1)(v + \lambda_2)$

$$= vu + \lambda_2 u + \lambda_1 v + \lambda_1\lambda_2 \quad , \quad \text{since} \quad uv = vu$$

$$= (v + \lambda_2)(u + \lambda_1) \quad .$$

Hence $T + \lambda$ is normal. The result 2.5.2 is easily seen to hold for pseudonormal operators $T \in B(X)$.

2.5.3 If $T \in B(X)$ is normal, then T^n is normal and spectro-normaloid for all non-negative integral n .

<u>Proof</u>:

Let $T = u + iv$ where $uv = vu$ and $u^s v^t$ are hermitian operators for all $s,t = 0,1,2,\ldots$.

$$T^n = (u + iv)^n$$

$$= P + iQ \quad ,$$

where P and Q are commuting polynomials in u and v with real coefficients. Also $P^M Q^N$ is a polynomial in u and v with real coefficients for $M,N = 0,1,2,\ldots$. Since each term $u^s v^t$ is hermitian for $s,t = 0,1,2,\ldots$, it follows that $P^M Q^N$ is hermitian for $M,N = 0,1,2,\ldots$. Thus T^n is normal. Now $T^n = P + iQ$ belongs to the closed subalgebra of $B(X)$ generated by u,v and the identity operator I , and since by 2.3 this closed subalgebra has property SN , it follows that T^n is spectro-normaloid.

2.5.4 If T is normal and X is reflexive then there exist operators $R,S \in B(X)$ such that

(i) $T = RS = SR$

(ii) $\sigma(R)$ is a set of non-negative real numbers and $\sigma(S)$ is a subset of the unit circle.

Proof:

By 2.5.1, since T is normal and X is reflexive, T is spectral of scalar type. Hence by 1.12 the result follows.

2.5.5 Lemma If X is reflexive, then an operator $T \in B(X)$ is hermitian if and only if T' , the adjoint of T , is hermitian.

Proof:

Let $T \in B(X)$ be hermitian. Then $W(T) = \{[Tx,x] : \|x\| = 1$ and $x \in X\}$ is real. Since X', the conjugate of X, is a Banach space we may introduce a semi-inner product on it. By the Hahn-Banach theorem in X', there exists for an arbitrary but fixed $f \in X'$, an $F_f \in X''$ such that

2.5.5.1 $$F_f(f) = \|f\|^2 \quad \text{and} \quad \|F_f\| = \|f\| \quad .$$

Thus we may define a semi-inner product $[\circ,\cdot]'$ on X' by requiring

2.5.5.2 $$[g,f]' = F_f(g) \quad , \qquad g \in X' \quad .$$

Let $\|f\| = 1$ and consider $[T'f,f]'$. By 2.5.5.2 we have

$$[T'f,f]' = F_f(T'f) \quad .$$

Since X is reflexive, there exists a unique $x_f \in X$ such that

$$F_f(T'f) = (T'f)(x_f)$$

and

$$\|x_f\| = \|F_f\| \quad .$$

Thus,

2.5.5.3 $$\begin{aligned} [T'f,f]' &= F_f(T'f) \\ &= (T'f)(x_f) \\ &= f(Tx_f) \quad , \end{aligned}$$

by the definition of T' .

Now x_f corresponds to F_f in the canonical mapping. Hence $F_f(f) = f(x_f)$. Thus by 2.5.5.1,

$$f(x_f) = F_f(f) = \|f\|^2 = \|F_f\|^2 = \|x_f\|^2$$

and

$$\|f\| = \|F_f\| = \|x_f\| \quad .$$

Hence

$$f(x_f) = \|x_f\|^2$$

and

$$\|f\| = \|x_f\| \quad .$$

Let us now define a semi-inner product $[\cdot,\circ]_1$ on X where we use the function f at the point x_f , that is such that

$$[y,x_f]_1 = f(y) \quad \text{for all} \quad y \in X \; .$$

Then by 2.5.5.3, we have

$$\begin{aligned} [T'f,f]' &= f(Tx_f) \\ &= [Tx_f,x_f]_1 \end{aligned}$$

Since $\|x_f\| = \|f\| = 1$, it follows that $[Tx_f,x_f]_1 \in W_1(T)$, the numerical range of T determined by the semi-inner product $[\circ,\cdot]_1$ on X . Since $W(T)$ is real, we have by 1.3.2 that any other determination $W_1(T)$ of the numerical range of T is real.

In particular, $[Tx_f, x_f]_1$ is real.

Hence $[T'f,f]'$ is real. Since $f \in X'$ is arbitrary with $\|f\| = 1$, it follows that the numerical range of T' determined by the semi-inner product $[\cdot,\cdot]'$ on X' is real. Thus T' is a hermitian operator on X'.

Suppose now that T' is a hermitian operator on X'. Then by what has been proved T'' is a hermitian operator on $X'' = X$, X being reflexive. Also $T'' = T$. Thus T is a hermitian operator on X.

Remark.

It follows from the property $\|T\| = \|T'\|$ and 1.4.1 that lemma 2.5.5 holds for non-reflexive Banach spaces also.

2.5.6 If X is reflexive, then an operator $T \in B(X)$ is normal if and only if T', the adjoint of T, is normal.

Proof:

Suppose T is normal and let $T = u + iv$ where $uv = vu$ and $u^m v^n$ are hermitian for all $m,n = 0,1,2,\ldots$. Now $T' = u' + iv'$ and

$$(uv)' = v'u' = (vu)' = u'v' \quad .$$

Since $u^m v^n$ are all hermitian we have by 2.5.5 that

$$(u^m v^n)' = u'^m \, v'^n$$

are hermitian for all $m,n = 0,1,2,\ldots$. Hence T' is normal.

Now suppose that T' is normal. Then by the immediately preceeding argument we have T'' normal. This implies, since $T'' = T$, that T is normal.

The result 2.5.6 clearly holds for pseudonormal operators $T \in B(X)$.

2.5.7 Fuglede's theorem

Let X be reflexive. Let $T \in B(X)$ be normal. Then any operator $B \in B(X)$ which commutes with T , commutes with $P(\Delta)$ for all $\Delta \in \mathcal{G}$ (see 1.9) , where $P(\cdot)$ is the resolution of the identity for T , (see [8]) .

Proof:

Since T is normal, it follows from 2.5.1 that it is spectral (of scalar type). Hence the result follows from 1.9.9.

2.5.8 Let X be reflexive. Let $S,T \in B(X)$ be normal and suppose that $S = u_s + iv_s$, $T = u_T + iv_T$ where $u_s v_s = v_s u_s$, $u_T v_T = v_T u_T$ and $u_s^m v_s^n$, $u_T^m v_T^n$ are hermitian operators for all $m,n = 0,1,2,\ldots$. If $ST = TS$, then $S + T$ is pseudonormal. If in addition $(u_s^m v_s^n)(u_T^{m'} v_T^{n'})$ are hermitian for $m,n,n',m' = 0,1,2,\ldots$, then $S + T$ and ST are both normal.

Proof:

Since S and T are normal, they are spectral (of scalar type) by 2.5.1 and so have resolutions of the identity $P_S(.)$ and $P_T(.)$ respectively. By Fuglede's theorem (2.5.7), since S is normal and commutes with T, T commutes with $P_S(.)$. Since $P_S(.)$ commutes with T and T is normal, it follows similarly that $P_S(.)$ commutes with $P_T(.)$. Thus

2.5.8.1 $$P_S(\cdot)\,P_T(\cdot) = P_T(\cdot)\,P_S(\cdot) \quad .$$

Now by the uniqueness (2.1) of u_S, v_S, u_T, v_T and the spectral representations (2.5.1):

$$T = \int_{\sigma(T)} \lambda\, P_T(d\lambda)\ , \qquad S = \int_{\sigma(S)} \mu\, P_S(d\mu)$$

we have

$$u_S = \int_{\sigma(S)} \mathrm{Re}\,\lambda\, P_S(d\lambda)\ , \qquad v_S = \int_{\sigma(S)} \mathrm{Im}\lambda\, P_S(d\lambda)$$

$$u_T = \int_{\sigma(T)} \mathrm{Re}\,\mu\, P_T(d\mu)\ , \qquad v_T = \int_{\sigma(T)} \mathrm{Im}\,\mu\, P_T(d\mu)$$

where "Re λ" and "Imλ" denote the real and imaginary parts of λ respectively.

Hence

$$u_s u_T = \int_{\sigma(s)} \operatorname{Re} \lambda \; P_s(d\lambda) \times \int_{\sigma(T)} \operatorname{Re} \mu \; P_T(d\mu) \quad .$$

Thus if $\pi_1 = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ and $\pi_2 = (\delta_1, \delta_2, \ldots, \delta_m)$ are partitions of $\sigma(s)$ and $\sigma(T)$ respectively with norms, $\|\pi_1\|$, $\|\pi_2\|$ (see 1.9.10), then

$$u_s u_T = \lim_{\|\pi_1\| \to o} \sum_{i=1}^{n} \operatorname{Re} \lambda_i P_s(\Delta_i) \times \lim_{\|\pi_2\| \to o} \sum_{j=1}^{m} \operatorname{Re} \mu_j \; P_T(\delta_j)$$

$$= \lim_{\|\pi_1\| \to o} \lim_{\|\pi_2\| \to o} \sum_{i,j=1}^{n,m} (\operatorname{Re} \lambda_i)(\operatorname{Re} \mu_j) \; P_s(\Delta_i) \; P_T(\delta_j) \quad .$$

Hence by 2.5.8.1, we have

$$u_s u_T = \lim_{\|\pi_1\| \to o} \lim_{\|\pi_2\| \to o} \sum_{i,j=1}^{n,m} (\operatorname{Re} \lambda_i)(\operatorname{Re} \mu_j) \; P_T(\delta_j) P_s(\Delta_i)$$

$$= \lim_{\|\pi_2\| \to o} \sum_{j=1}^{m} \operatorname{Re} \mu_j \; P_T(\delta_j) \times \lim_{\|\pi_1\| \to o} \sum_{i=1}^{n} \operatorname{Re} \lambda_i \; P_s(\Delta_i)$$

$$= \int_{\sigma(T)} \operatorname{Re} \mu \; P_T(d\mu) \times \int_{\sigma(s)} \operatorname{Re} \lambda \; P_s(d\lambda)$$

$$= u_T \, u_s \quad .$$

Similarly, $u_s v_T = v_T u_s$, $v_s u_T = u_T v_s$, $v_s v_T = v_T v_s$.

Thus we have

2.5.8.2
$$u_s u_T = u_T u_s$$
$$u_s v_T = v_T u_s$$
$$v_s u_T = u_T v_s$$
$$v_s v_T = v_T v_s$$

Now $S + T = (u_s + u_T) + i(v_s + v_T)$. Thus by 2.5.8.2 we have

$$\begin{aligned} &(u_s + u_T)(v_s + v_T) \\ &= u_s v_s + u_s v_T + u_T v_s + u_T v_T \\ &= v_s u_s + v_T u_s + v_s u_T + v_T u_T \\ &= (v_s + v_T)(u_s + u_T) \quad . \end{aligned}$$

Since u_s, u_T, v_s, v_T are hermitian operators, so are $u_s + u_T$ and $v_s + v_T$. Thus $S + T$ is pseudonormal.

For all $M,N = 0,1,2,\ldots,$ $(u_s + u_T)^M(v_s + v_T)^N$ is the sum of real multiples of terms of the form

$$\begin{aligned} &u_s^m u_T^n v_s^{m'} v_T^{n'} \\ &= (u_s^m v_s^{m'})(u_T^n v_T^{n'}). \end{aligned}$$

Since by hypothesis $(u_s^m v_s^{m'})(u_T^n v_T^{n'})$ is hermitian for all $m,m',n,n' = 0,1,2,\ldots$, it follows that $(u_s + u_T)^M (v_s + v_T)^N$ is also hermitian for all $M,N = 0,1,2,\ldots$. Hence $S + T$ is normal.

$$ST = (u_s u_T - v_s v_T) + i(u_s v_T + v_s u_T)$$

Also $(u_s u_T - v_s v_T)$ and $(u_s v_T + v_s u_T)$ commute on account of 2.5.8.2 . Now for $M,N = 0,1,2,\ldots$, $(u_s u_T - v_s v_T)^M (u_s v_T + v_s u_T)^N$ is the sum of real multiples of terms of the form

$$(u_s^m v_s^n)(u_T^{m'} v_T^{n'}) \quad .$$

Thus by our hypothesis,

$$(u_s u_T - v_s v_T)^M (u_s v_T + v_s u_T)^N$$

is hermitian for all $M,N = 0,1,2,\ldots$. Hence ST is normal.

2.5.9 Lemma Let $T \in B(X)$. If $T = u + iv$ where u and v are hermitian operators in $B(X)$ and M is one of a pair (M,N) of subspaces of X which reduces T completely, then

$$T|_M = u|_M + iv|_M \quad ,$$

where $u|_M$ and $v|_M$ are both hermitian operators in $B(M)$.

Proof:

By 1.5,

$$X = M \oplus N \quad .$$

Also M and N are both invariant under T and so $T|_M$ and $T|_N$ are both defined. For arbitrary $x \in X$, we can write

$$x = x_1 + x_2 \, , \quad \text{where} \quad x_1 \in M \, , \quad x_2 \in N \quad .$$

Hence $Tx = Tx_1 + Tx_2$ and so, since $x_1 \in M$ and $x_2 \in N$ we have

$$ux + ivx = T|_M x_1 + T|_N x_2 \quad .$$

If $x \in M$, then $x_1 = x$ and $x_2 = 0$ and so

$$\begin{aligned} ux + ivx &= T|_M x + T|_N 0 \\ &= T|_M x \quad . \end{aligned}$$

Thus

$$u|_M x + iv|_M x = T|_M x \quad .$$

This holds for arbitrary $x \in M$, and so

$$T|_M = u|_M + iv|_M \quad .$$

Now $W\{u|_M\}$ and $W\{v|_M\}$ are subsets respectively of $W(u)$ and $W(v)$ and since both of these are real on account of the hermiticity of u

and v , it follows that $W\{u|_M\}$ and $W\{v|_M\}$ are both real.

Thus $T|_M = u|_M + iv|_M$, where $u|_M$ and $v|_M$ are hermitian operators in $B(M)$.

2.5.10 If $T \in B(X)$ is normal and M is one of a pair of subspaces of X which reduces T completely, then $T|_M$ is also normal.

Proof:

Let $T = u + iv$, where $uv = vu$ and $u^m v^n$ are hermitian operators for $m,n = 0,1,2,\ldots$.

By 2.5.9, we have

$$T|_M = u|_M + iv|_M \text{ , where } u|_M \text{ and } v|_M$$

are hermitian operators in $B(M)$. Since $uv = vu$, we have

$$(uv)|_M = (vu)|_M \quad .$$

Thus $u|_M \, v|_M = v|_M \, u|_M$.

Also $(u^m v^m)|_M = (u|_M)^m \, (v|_M)^n$ are hermitian for $m,n = 0,1,2,\ldots$.

Hence $T|_M$ is normal.

2.5.11 The limit in the uniform operator topology of a sequence of normal operators is normal.

Proof:

Let T_n be normal for each n and suppose that $T_n \to T$. We can write $T_n = u_n + iv_n$ where $u_n v_n = v_n u_n$ and $u_n^s v_n^t$ are hermitian for all $s,t = 0,1,2,\ldots$.

By an argument similar to that used in the proof of lemma 2.3.3, u_n and v_n converge to u_o and v_o which commute. Also $T = u_o + iv_o$. Since $u_n^s v_n^t$ is hermitian, it follows that

$$u_o^s v_o^t = \lim_{n \to \infty} u_n^s v_n^t$$

is hermitian for all $s,t = 0,1,2,\ldots$. Thus T is normal.

CHAPTER III

HYPONORMAL OPERATORS

3.1 Definition

An operator $T \in B(X)$ will be called hyponormal if and only if.

3.1.1 T can be written in the form $T = u + iv$, where u and $v \in B(X)$ and are hermitian operators.

3.1.2 $W\{i(uv - vu)\} \geq 0$, where W is the numerical range with respect to some semi-inner product on X.

3.1.3 For any subspace M of X which is one of a pair that reduces T completely, we have $(T - \lambda I)|_M$ is spectronormaloid for $\lambda = 0$ or $\lambda \in \sigma(T)$ the spectrum of T.

3.2 Some comments on the definition

3.2.1 The condition 3.1.2 always makes sense because I. Vidav [19, p. 123] has shown that if u and v are hermitian then so is $i(uv - vu)$.

3.2.2 Since a Banach space can in general be made into a semi-

inner product space by the introduction of any one of infinitely many semi-inner products on X , we must show that the condition 3.1.2 is independent of the semi-inner product we use on X . Let $T = u + iv$, where u and v are hermitian operators in $B(X)$, satisfy 3.1.2 with respect to some semi-inner product $[\cdot,\cdot]$ on X . Then

$$W\{i(uv-vu)\} = \{[i(uv-vu)x,x] : \|x\| = 1\} \geq 0 \quad .$$

Let $[\cdot,\cdot]_1$ be any other semi-inner product on X and let

$$W_1\{i(uv - vu)\} = \{[i(uv - vu)x,x]_1 : \|x\| = 1\} \quad .$$

By 1.3.1, we have

$$\mathcal{C}W_1\{i(uv - vu)\} = \mathcal{C}\, W\{i(uv - vu)\} \geq 0 \quad ,$$

where $\mathcal{C}$ denotes convex hull.

Hence $W_1\{i(uv - vu)\} \geq 0$.

Thus if T satisfies 3.1.2 for one semi-inner product on X , then it satisfies it for any other semi-inner product.

3.2.3 If X is a Hilbert space, the condition 3.1.2 is equivalent to the condition $T^*T - TT^* \geq 0$.

Proof:

Since X is a Hilbert space, there is a unique semi-inner product on X (see 1.2.5) namely the inner product. Thus condition 3.1.2 is now

$$3.2.3.1 \qquad \{(i[uv - vu]x,x) : \|x\| = 1\} \geq 0 .$$

Let $z \in X$ be arbitrary. If $z = 0$, then we have

$$3.2.3.2 \qquad (i[uv - vu]z,z) = 0 .$$

If $z \neq 0$, it is easily verified that 3.2.3.1 leads to

$$(i[uv - vu]z,z) \geq 0 .$$

Thus by 3.2.3.2 we have

$$(i[uv - vu]x,x) \geq 0 \quad \text{for all} \quad x \in X .$$

Conversely if $(i[uv - vu]x,x) \geq 0$ for all $x \in X$, then $(i[uv - vu]x,x) \geq 0$ for all x such that $\|x\| = 1$.

That is $\{(i[uv - vu]x,x) : \|x\| = 1\} \geq 0$. Thus if X is a Hilbert space, the condition 3.1.2 is equivalent to

$$3.2.3.3 \qquad \{(i[uv - vu]x,x) : x \in X\} \geq 0 .$$

Since X is a Hilbert space, every $T \in B(X)$ can be written $T = u + iv$,

where u and $v \in B(X)$ and are hermitian operators. Then $T^* = u - iv$. Hence $T^*T - TT^* = 2i(uv - vu)$. Thus since T satisfies 3.1.2 if and only if it satisfies 3.2.3.3, we have

$$\{([T^*T - TT^*]x,x) : x \in X\} \geq 0 \quad .$$

Hence $T^*T - TT^* \geq 0$.

3.2.4 If X is a Hilbert space, then condition 3.1.2 implies condition 3.1.3.

Proof:

By 3.2.3, the condition 3.1.2 is equivalent to $T^*T - TT^* \geq 0$. This condition in a Hilbert space is equivalent to the hyponormality of T according to the usual Hilbert space definition of hyponormality, (see 1.14.1).

By 1.14.2, $T - \lambda$ is thus hyponormal for all complex λ and hence in particular for $\lambda = 0$ or $\lambda \in \sigma(T)$.

If M is one of a pair of subspaces of X which reduces T completely then M is invariant under $T - \lambda$ and since $T - \lambda$ is hyponormal, we have by 1.14.1 that $(T - \lambda)|_M$ is hyponormal. Hence by 1.14.3,

$$\|(T - \lambda)|_M\| = r[(T - \lambda)|_M] \quad .$$

The condition 3.1.2 thus implies 3.1.3.

3.2.5 If X is a Hilbert space, then the definition of hyponormality in 3.1 is equivalent to the usual Hilbert space definition.

Proof:

This follows readily from 3.2.3 and 3.2.4.

3.3 Some properties of hyponormal operators

3.3.1 If $T \in B(X)$ and is hyponormal, then so is λT for all complex λ. If $\lambda \in \sigma(T)$, then $T - \lambda$ is also hyponormal.

Proof:

Let $T = u + iv$, where u and v are hermitian operators in $B(X)$ and $W\{i(uv - vu)\} \geq 0$.

Let $\lambda = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real. Then

$$\begin{aligned} \lambda T &= (\lambda_1 + i\lambda_2)(u + iv) \\ &= (\lambda_1 u - \lambda_2 v) + i(\lambda_1 v + \lambda_2 u) \quad . \end{aligned}$$

Since u and v are hermitian operators, so are $\lambda_1 u - \lambda_2 v$ and $\lambda_1 v + \lambda_2 u$. Now

$$W\{i[(\lambda_1 u - \lambda_2 v)(\lambda_1 v + \lambda_2 u) - (\lambda_1 v + \lambda_2 u)(\lambda_1 u - \lambda_2 v)]\}$$

$$= W\{i[\lambda_1^2 + \lambda_2^2)(uv - vu)]\}$$

$$= (\lambda_1^2 + \lambda_2^2)\ W\{i(uv - vu)\} \geq 0 \quad .$$

Since T is hyponormal, we have by definition that if M is any one of a pair of subspaces that reduces T completely, then

$$\|(T - \lambda_o)|_M\| = r\{(T - \lambda_o)|_M\} \quad \text{for} \quad \lambda_o = 0 \quad \text{or} \quad \lambda_o \in \sigma(T) \quad .$$

Let $\mu = 0$ or $\mu \in \sigma(\lambda T)$.

Then we can write $\mu = \lambda\lambda_o$. Hence

$$\begin{aligned} r\{(\lambda T - \mu)|_M\} &= r\{(\lambda T - \lambda\lambda_o)|_M\} \\ &= |\lambda|\ r\{(T - \lambda_o)|_M\} \\ &= |\lambda| \cdot \|(T - \lambda_o)|_M\| \\ &= \|(\lambda T - \lambda\lambda_o)|_M\| \\ &= \|(\lambda T - \mu)|_M\| \quad . \end{aligned}$$

Thus λT is hyponormal.

Now $T - \lambda = (u - \lambda_1) + i(v - \lambda_2)$. Clearly $u - \lambda_1$ and $v - \lambda_2$ are hermitian operators and

$$W\{i[(u-\lambda_1)(v - \lambda_2) - (v - \lambda_2)(u - \lambda_1)]\}$$
$$= W\{i(uv - vu)\} \geq 0 \quad .$$

Let $\mu = 0$ or $\mu \in \sigma(T - \lambda)$. Then since $\lambda \in \sigma(T)$ we can write

$$\mu = \lambda_o - \lambda \ , \quad \text{where} \quad \lambda_o \in \sigma(T) \quad .$$

Thus $[(T - \lambda) - \mu]|_M = [T - \lambda - (\lambda_o - \lambda)]|_M$

$$= (T - \lambda_o)|_M \quad .$$

Hence, since T is hyponormal and $\lambda_o \in \sigma(T)$, we have by 3.1.3 that

$$r\{[(T - \lambda) - \mu]|_M\} = \|[(T - \lambda) - \mu]|_M\|$$

for $\mu = 0$ or $\mu \in \sigma(T - \lambda)$.

This shows that $T - \lambda$ is hyponormal.

3.3.2 If $T \in B(X)$ and is normal, then it is hyponormal.

Proof:

Let $T = u + iv$ where $uv = vu$ and $u^m v^n$ are hermitian operators for $m,n = 0,1,2,\ldots$.

Then $W\{i(uv - vu)\} = W\{0\} = 0$.

Since T is normal, so is $T + \lambda$ for all complex λ (see 2.5.2). In particular $T - \lambda_o$ is normal for all $\lambda_o = 0$ or $\lambda_o \in \sigma(T)$. Thus

by 2.5.10, if M is any one of a pair of subspaces of X which reduces T completely, then $(T - \lambda_o)|_M$ is also normal. Hence by 2.5.3, $(T - \lambda_o)|_M$ is spectro-normaloid.

Thus T is hyponormal.

3.3.3 If $T \in B(X)$ and is hyponormal and M is any one of a pair of subspaces of X which reduces T completely, then $T|_M$ is hyponormal.

Proof:

Since T is hyponormal, we can write $T = u + iv$, where u and v are hermitian operators in $B(X)$ such that

$$W\{i(uv - vu)\} \geq 0 .$$

Since M is one of a pair of subspaces that reduces T completely, we have by lemma 2.5.9, that

$$T|_M = u|_M + iv|_M \quad \text{where} \quad u|_M \text{ and } v|_M$$

are hermitian opeators in $B(M)$.

Now $\quad W\{i(u|_M\, v|_M - v|_M\, u|_M)\}$

$$= W\{i(uv - vu)|_M\} \geq 0$$

since $W\{i(uv - vu)|_M\}$ is a subset of $W\{i(uv - vu)\}$.

Let $N \subset M$ be any one of a pair of subspaces of M which reduces $T|_M$

completely. Then clearly N is one of a pair of subspaces of X which reduces T completely. Also for any λ,

$$(T - \lambda)|_N = \{(T - \lambda)|_M\}|_N \quad .$$

Since T is hyponormal, $(T - \lambda)|_N$ is spectro-normaloid for $\lambda = 0$ or $\lambda \in \sigma(T)$. Hence $(T|_M - \lambda)|_N$ is spectro-normaloid for $\lambda = 0$ or $\lambda \in \sigma(T)$. Thus in particular $(T|_M - \lambda)|_N$ is spectro-normaloid for $\lambda = 0$ or $\lambda \in \sigma(T|_M) \subseteq \sigma(T)$. Thus $T|_M$ is hyponormal.

3.3.4 The only generalised quasi-nilpotent hyponormal operator in $B(X)$ is the zero operator.

<u>Proof</u>:

Let T be a generalised quasi-nilpotent hyponormal operator in $B(X)$. Then $r(T) = 0$.

Since T is hyponormal, we have by 3.1.3 that if M is any one of a pair of subspaces of X which reduces T completely then $(T - \lambda)|_M$ is spectro-normaloid for $\lambda = 0$ or $\lambda \in \sigma(T)$. Let us choose $M = X$ and $\lambda = 0$.

Then we have

$$\|T\| = r(T) = 0 \quad .$$

Thus $T = 0$.

3.3.5 Let T be a hyponormal operator in $B(X)$. If λ_o is an isolated point of $\sigma(T)$, then $\lambda_o \in P\sigma(T)$. Also the subspace $M(\lambda_o) = \{x \in X : Tx = \lambda_o x\}$ is one of a pair which reduces T completely, and $T|_{M(\lambda_o)}$ is normal.

<u>Proof</u>:

Since λ_o is an isolated part of $\sigma(T)$, it follows by the decomposition theorem (see 1.6) that there exists a subspace $M_{\{\lambda_o\}}$ of X which is one of a pair which reduces T completely and is such that

$$\sigma(T|_{M_{\{\lambda_o\}}}) = \{\lambda_o\} \quad .$$

Also by 3.3.3, $T|_{M_{\{\lambda_o\}}}$ is hyponormal. Hence $T|_{M_{\{\lambda_o\}}} - \lambda_o$ is hyponormal by 3.3.1 since

$$\lambda_o \in \sigma(T|_{M_{\{\lambda_o\}}}) \quad .$$

Now

$$\begin{aligned} &\sigma(T|_{M_{\{\lambda_o\}}} - \lambda_o) \\ &\quad = \{\sigma(T|_{M_{\{\lambda_o\}}}) - \lambda_o\} \\ &\quad = \{\lambda_o - \lambda_o\} \\ &\quad = \{0\} \quad . \end{aligned}$$

Hence

$$r\{T|_{M_{\{\lambda_o\}}} - \lambda_o\} = 0 \quad .$$

Thus $T|_{M_{\{\lambda_o\}}} - \lambda_o$ is a generalised quasi-nilpotent hyponormal operator.

Hence by 3.3.4,

$$T|_{M_{\{\lambda_o\}}} - \lambda_o \text{ is the zero operator in } B(M_{\{\lambda_o\}}) \quad .$$

Thus

$$T|_{M_{\{\lambda_o\}}} x = \lambda_o x \quad \text{for all} \quad x \in M_{\{\lambda_o\}} \quad .$$

Using theorem 1.6 we in fact have,

$$M_{\{\lambda_o\}} = \{x \in X : Tx = \lambda_o x\} = M(\lambda_o) \quad .$$

Since by 1.6 $M_{\{\lambda_o\}}$ is non-trivial, there is some $x \neq 0$ in $M(\lambda_o)$.

Hence $\lambda_o \in P\sigma(T)$.

Since T is hyponormal, we can write $T = u + iv$ where u and v are hermitian operators in $B(X)$.

Since $M(\lambda_o)$ is one of a pair of subspaces which reduces T completely we have by lemma 2.5.9, that

$$T|_{M(\lambda_o)} = u|_{M(\lambda_o)} + iv|_{M(\lambda_o)}$$

where $u|_{M(\lambda_o)}$ and $v|_{M(\lambda_o)}$ are hermitian operators in $B[M(\lambda_o)]$.

For $\|x\| = 1$ and $x \in M(\lambda_o)$ we have

$$[T|_{M(\lambda_o)}x,x] = [u|_{M(\lambda_o)}x,x] + i[v|_{M(\lambda_o)}x,x] \quad .$$

Hence

$$[\lambda_o x,x] = [u|_{M(\lambda_o)}x,x] + i[v|_{M(\lambda_o)}x,x] \ , \quad \text{since} \quad T|_{M(\lambda_o)} = \lambda_o \quad .$$

Let $\lambda_o = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real. Then

$$[(u|_{M(\lambda_o)} - \lambda_1)x,x] = i[(\lambda_2 - v|_{M(\lambda_o)})x,x] \quad .$$

But $u|_{M(\lambda_o)} - \lambda_1$, and $\lambda_2 - v|_{M(\lambda_o)}$ are both hermitian operators in $B[M(\lambda_o)]$ and so

$$[(u|_{M(\lambda_o)} - \lambda_1)x,x] \quad \text{and} \quad [(\lambda_2 - v|_{M(\lambda_o)})x,x]$$

are both real.

It follows then that

$$[(u|_{M(\lambda_o)} - \lambda_1)x,x] = 0 = [(\lambda_2 - v|_{M(\lambda_o)})x,x] \quad .$$

But by [2, p. 336],

$$\|u|_{M(\lambda_o)} - \lambda_1\| \leq 4 \sup \{|\lambda| : \lambda \in W[u|_{M(\lambda_o)} - \lambda_1]\}$$

Hence

$$\|u|_{M(\lambda_o)} - \lambda_1\| = 0 \quad .$$

That is

$$u|_{M(\lambda_o)} = \lambda_1 I|_{M(\lambda_o)} .$$

Similarly,

$$v|_{M(\lambda_o)} = \lambda_2 I|_{M(\lambda_o)} .$$

Hence

$$u|_{M(\lambda_o)} \, v|_{M(\lambda_o)} = \lambda_1\lambda_2 I|_{M(\lambda_o)} = v|_{M(\lambda_o)} \, u|_{M(\lambda_o)} .$$

Also

$$\{u|_{M(\lambda_o)}\}^m \, \{v|_{M(\lambda_o)}\}^n = \lambda_1^m \, \lambda_2^n \, I|_{M(\lambda_o)}$$

is a hermitian operator in $B[M(\lambda_o)]$ for all $m,n = 0,1,2,\ldots$.

Thus $T|_{M(\lambda_o)}$ is normal.

3.3.6 Let T be a non-zero hyponormal operator in $B(X)$. If $\sigma(T)$ contains exactly one cluster point, then T is normal.

Proof:

Without loss of generality we may assume that the single cluster point in $\sigma(T)$ is 0 , for if the cluster point $\mu_o \neq 0$, we may consider in place of T , the operator $T - \mu_o$ which is also hyponormal by 3.3.1 and has 0 as the single cluster point in its spectrum. Since T is hyponormal, it is spectro-normaloid by 3.1.3.

Hence

$$\|T\| = r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\} \quad .$$

Let

$$|\lambda_1| = \max \{|\lambda| : \lambda \in \sigma(T)\} \quad .$$

Then

$$|\lambda_1| = \|T\| \neq 0 \quad .$$

Since $\lambda_1 \neq 0$, it is an isolated point of $\sigma(T)$. Hence by 3.3.5, $\lambda_1 \in P\sigma(T)$.

Let $M_1 = \{x \in X : Tx = \lambda_1 x\}$.

Then since $\lambda_1 \in P\sigma(T)$, M_1 is non-trivial. Furthermore, by 3.3.5, M_1 is one of a pair of subspaces of X which reduces T completely and $T|_{M_1}$ is normal. Thus if N_1 is the other member of the pair of which M_1 is one, then by 1.5 we also have

$$X = M_1 \oplus N_1 \quad .$$

Hence by 1.6,

$$\sigma(T|_{N_1}) = \sigma(T) - \{\lambda_1\} \quad .$$

Let

$$|\lambda_2| = \max\{|\lambda| : \lambda \in \sigma(T|_{N_1})\} \quad .$$

Then

$$|\lambda_2| \leq |\lambda_1| \quad .$$

Also N_1 is one of a pair of subspaces which reduces T, and so by 3.3.3, $T|_{N_1}$ is hyponormal. Thus

$$\|T|_{N_1}\| = r(T|_{N_1}) = |\lambda_2| \quad .$$

Define

$$M_2 = \{x \in N_1 : T|_{N_1} x = \lambda_2 x\}$$

$$= \{x \in X : Tx = \lambda_2 x\} \quad .$$

Applying 3.3.5 again, M_2 is one of a pair of subspace (M_2, N_2) of N_1 which reduces $T|_{N_1}$ completely and $(T|_{N_1})|_{M_2} = T|_{M_2}$ is normal. Thus we have $X = M_1 \oplus M_2 \oplus N_2$. Also by 3.3.3, $(T|_{N_1})|_{N_2} = T|_{N_2}$ is hyponormal and by 1.6,

$$\sigma(T|_{N_2}) = \sigma(T|_{N_1}) - \{\lambda_2\}$$

$$= \sigma(T) - \{\lambda_1\} \cup \{\lambda_2\} \quad .$$

Let

$$|\lambda_3| = \max \{|\lambda| : \lambda \in \sigma(T|_{N_2})\} \quad .$$

Then

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \quad .$$

Define

$$M_3 = \{x \in N_2 : T|_{N_2} x = \lambda_3 x\}$$
$$= \{x \in X : Tx = \lambda_3 x\} \quad , \text{ by } 1.6 \ .$$

We now repeat the argument used twice already.

At the n^{th} step we shall obtain $X = M_1 \oplus M_2 \oplus \ldots \oplus M_n \oplus N_n$, where $T|_{M_i}$ is normal for each i and $T|_{N_n}$ is hyponormal with

$$\|T|_{N_n}\| = |\lambda_{n+1}| = \max\{|\lambda| : \lambda \in \sigma(T|_{N_n})\} \ .$$

Also

$$\|T\| = |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_{n+1}| > 0 \ .$$

Thus $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence in $\sigma(T)$, a compact set, and so contains a convergent subsequence. This subsequence can only converge to 0. The fact that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| \geq \ldots$ then implies that $\{\lambda_n\}_{n=1}^{\infty}$ converges to 0, since 0 is the only cluster point in $\sigma(T)$. Thus $\lambda_n \to 0$ as $n \to \infty$.

Since $T|_{M_i}$ is normal for each i, so is $T|_{M_1 \oplus M_2 \oplus \ldots \oplus M_n}$.
Let us define an operator $T_n : X \to X$ where

$$T_n x = \begin{cases} T|_{M_1 \oplus M_2 \oplus \ldots \oplus M_n} x & \text{if } x \in M_1 \oplus M_2 \oplus \ldots \oplus M_n \\ 0 , & \text{if } x \notin M_1 \oplus M_2 \oplus \ldots \oplus M_n \ . \end{cases}$$

Let us also define an operator $T_n^{\#} : X \to X$ where

$$T_n^{\#} x = \begin{cases} T|_{N_n} x & \text{if} \quad x \in N_n \\ 0 , & \text{if} \quad x \notin N_n \end{cases} .$$

Now

$$\begin{aligned} \|T_n^{\#}\| &= \sup \{ \|T_n^{\#} x\| : x \in X \text{ and } \|x\| = 1 \} \\ &= \sup \{ \|T|_{N_n} x\| : x \in N_n \text{ and } \|x\| = 1 \} \\ &= \|T|_{N_n}\| . \end{aligned}$$

Also

$$\begin{aligned} \|T_n - T\| &= \sup \{ \|T_n x - Tx\| : x \in X \text{ and } \|x\| = 1 \} \\ &= \sup \{ \|T_n x - Tx\| : x \notin M_1 \oplus M_2 \oplus \ldots \oplus M_n \text{ and } \|x\| = 1 \} \end{aligned}$$

Let $x \in X$. Then $x = y_1 + y_2$, where $y_1 \in M_1 \oplus M_2 \oplus \ldots \oplus M_n$ and $y_2 \in N_n$. Now $T_n^{\#} x = T_n^{\#} y_1 + T_n^{\#} y_2$

$$= T|_{N_n} y_2 .$$

Also $T_n x = T_n y_1 + T_n y_2$

$$= T_n y_1 .$$

Hence

$$\begin{aligned} T_n^{\#} x + T_n x &= T|_{N_n} y_2 + T_n y_1 \\ &= Ty_2 + Ty_1 \\ &= Tx . \end{aligned}$$

Hence $T_n x - Tx = -T_n^{\#} x$ and so

$$\begin{aligned}
\|T_n - T\| &= \sup\{\|T_n^{\#} x\| : x \notin M_1 \oplus M_2 \oplus \dots \oplus M_n \text{ and } \|x\| = 1\} \\
&\leq \sup\{\|T_n^{\#}\| \cdot \|x\| : x \notin M_1 \oplus M_2 \oplus \dots \oplus M_n \text{ and } \|x\| = 1\} \\
&= \|T_n^{\#}\| \\
&= \|T|_{N_n}\| \\
&= |\lambda_{n+1}| \quad .
\end{aligned}$$

Hence

$$\lim_{n \to \infty} \|T_n - T\| = \lim_{n \to \infty} |\lambda_{n+1}| = 0 \quad .$$

Thus T is the limit in the uniform operator topology of the sequence $\{T_n\}_{n=1}^{\infty}$ of normal operators in $B(X)$. It follows by 2.5.11, that T is normal.

Corollary I

If T is a hyponormal completely continuous operator in $B(X)$ where X is infinite dimensional and $P\sigma(T)$ is infinite, then T is normal.

Proof:

Since T is completely continuous and X is infinite dimensional, we have by theorem 5.5-F and 5.5-G in [18, p. 281-282] that

(i) $0 \in \sigma(T)$

(ii) $P\sigma(T)$ contains at most a countable set of points with 0 as the only possible cluster point.

Since by hypothesis $P\sigma(T)$ is infinite, we have by (i) and (ii) and the compactness of $\sigma(T)$ that 0 is the one cluster point of $\sigma(T)$. Thus by 3.3.6, T is normal.

Corollary II

If $T \in B(X)$ is hyponormal, where X is finite dimensional and $0 \in \sigma(T)$ then T is normal.

Proof:

With the same notation as in the proof of 3.3.6, we have

$$\|T_n - T\| \leq |\lambda_{n+1}| \quad \text{for some } n \quad .$$

Since X is finite dimensional, every operator $T \in B(X)$ is completely continuous [see p. 285, problem 1, 18].

Also $P\sigma(T) = \sigma(T)$. Since $0 \in \sigma(T) = P\sigma(T)$, there is an n such that $\lambda_{n+1} = 0$. For this n, we have

$$\|T_n - T\| = 0 \ .$$

Hence $T = T_n$.

Since T_n is normal, it follows that T is also normal.

3.3.7 Not every hyponormal operator is normal.

Proof:

We shall exhibit a counter example. Take $x = \ell'(n)$, $n \geq 3$, then the matrix

$$C = \left\| \begin{matrix} c_{11}, & c_{12}, & \cdots & c_{1n} \\ c_{21} & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ c_{n1}, & & & c_{nn} \end{matrix} \right\|$$

is a linear operator on $\ell'(n)$ and

$$\|C\| = \max_{1 \leq \beta \leq n} \left\{ \sum_{\alpha=1}^{n} |c_{\alpha\beta}| \right\} .$$

Let

$$s_\beta = \sum_{\alpha=1}^{n} c_{\alpha\beta} ,$$

where the $c_{\alpha\beta}$ are non-negative real numbers. Consider the matrix

$$M = (c_{\alpha\beta} + i\delta_{\alpha\beta}[s_\beta - c_{\beta\beta}]) ,$$

whose $(\alpha,\beta)^{th}$ element is $c_{\alpha\beta} + i\delta_{\alpha\beta}[s_\beta - c_{\beta\beta}]$ where $i = \sqrt{-1}$ and $\delta_{\alpha\beta}$ is the Kronecker delta.

Let t be a real, positive, small number. Then if I is the identity operator on $\ell'(n)$, we have

$$\|I + it\,M\| = \|(\delta_{\alpha\beta} + it[c_{\alpha\beta} + i\delta_{\alpha\beta}\{s_\beta - c_{\beta\beta}\}])\|$$

$$= \max_{1 \le \beta \le n} \left\{ \sum_{\alpha=1}^{n} |\delta_{\alpha\beta} + it[c_{\alpha\beta} + i\delta_{\alpha\beta}\{s_\beta - c_{\beta\beta}\}]| \right. .$$

Now

$$\sum_{\alpha=1}^{n} |\delta_{\alpha\beta} + it[c_{\alpha\beta} + i\delta_{\alpha\beta}(s_\beta - c_{\beta\beta})]|$$

$$= \sum_{\substack{\alpha=1 \\ \alpha\neq\beta}}^{n} |itc_{\alpha\beta}| + |1 + it[c_{\beta\beta} + i(s_\beta - c_{\beta\beta})]|$$

$$= t(s_\beta - c_{\beta\beta}) + \sqrt{[1-t(s_\beta - c_{\beta\beta})]^2 + t^2 c_{\beta\beta}^2}$$

$$= t(s_\beta - c_{\beta\beta}) + \sqrt{1-2t(s_\beta - c_{\beta\beta}) + t^2\{c_{\beta\beta}^2 + (s_\beta - c_{\beta\beta})^2\}}$$

$$= t(s_\beta - c_{\beta\beta}) + \sqrt{1-2t\{s_\beta - c_{\beta\beta} - t\tfrac{L}{2}\}} ,$$

where $L = c_{\beta\beta}^2 + (s_\beta - c_{\beta\beta})^2$.

$$= t(s_\beta - c_{\beta\beta}) + 1 + \frac{1}{2}\left[-2t\{s_\beta - c_{\beta\beta} - t\,\frac{1}{2}\}\right]$$

$$+ \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)}{2!}\;4t^2\{s_\beta - c_{\beta\beta} - t\,\frac{1}{2}\}^2 + \ldots$$

$$= 1 + o(t) \quad \text{as} \quad t \to 0\ , \quad \text{for all} \quad \beta\ .$$

Thus

$$\max_{1 \le \beta \le n} \left\{ \sum_{\alpha=1}^{n} |\delta_{\alpha\beta} + it[c_{\alpha\beta} + i\delta_{\alpha\beta}\,(s_\beta - c_{\beta\beta})]| \right\}$$

$$= 1 + o(t) \quad \text{as} \quad t \to 0\ .$$

Thus $\|I + it\,M\| = 1 + o(t)$ as $t \to 0$. Hence by 1.4.1, M is a hermitian operator in $\ell'(n)$.

Written out in the usual way M is

$$\left\| \begin{matrix} c_{11} + i\sum_{\alpha=2}^{n} c_{\alpha 1}\ , & c_{12}\ , \\ c_{21}\ , & c_{22} + i\sum_{\substack{\alpha=1 \\ \alpha\neq 2}}^{n} c_{\alpha 2}\ , \\ c_{31}\ , & c_{32}\ , \\ \vdots & \\ c_{n1} & \end{matrix} \right\|$$

It is easily checked that the matrix obtained from M and shown below is also a hermitian operator on $\ell'(n)$:

$$N = \left\| \begin{array}{lll} c_{11} + i \sum\limits_{\alpha=2}^{n} c_{\alpha 1} , & ic_{12} & , \\ ic_{21} & , \; c_{22} + i \sum\limits_{\substack{\alpha=1 \\ \alpha \neq 2}}^{n} c_{\alpha 2} & , \\ ic_{31} & , \; ic_{32} & , \\ ic_{41} & , \; ic_{42} & , \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{array} \right\|$$

Now we consider the operator $T = M + iN$ on $\ell'(n)$. We have

$$M + iN = \left\| \begin{array}{lll} (c_{11} + i \sum\limits_{\alpha=2}^{n} c_{\alpha 1})(1+i) , & & \bigcirc \\ & , \; (c_{22} + i \sum\limits_{\substack{\alpha=1 \\ \alpha \neq 2}}^{n} c_{\alpha 2})(1+i) , & \\ & \bigcirc & \\ & & \end{array} \right\|$$

where the off-diagonal elements all vanish.

Let us define

$$C_{\beta\beta} = c_{\beta\beta} + i \sum_{\substack{\alpha=1 \\ \alpha\neq\beta}}^{n} c_{\alpha\beta}$$

$$C_{\alpha\beta} = c_{\alpha\beta} , \quad \text{for} \quad \alpha \neq \beta \quad .$$

Then $M = (C_{\alpha\beta})$ and

$$N = \left\| \begin{matrix} C_{11}, & iC_{12} , & \\ iC_{21}, & C_{22} , & \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{matrix} \right\|$$

Hence the $(\alpha,\beta)^{th}$ element in MN is

$$\sum_{\substack{r=1 \\ r\neq\beta}}^{n} iC_{\alpha r}C_{r\beta} + C_{\alpha\beta}C_{\beta\beta} \quad .$$

Also the $(\alpha,\beta)^{th}$ element in NM is

$$\sum_{\substack{r=1 \\ r \neq \alpha}}^{n} iC_{\alpha r}C_{r\beta} + C_{\alpha\alpha}C_{\alpha\beta} \quad .$$

Hence the $(\alpha,\beta)^{th}$ element in $MN - NM$ is

$$i \sum_{r=1}^{n} C_{\alpha r}C_{r\beta} - iC_{\alpha\beta}C_{\beta\beta} + C_{\alpha\beta}C_{\beta\beta}$$

$$- i \sum_{r=1}^{n} C_{\alpha r}C_{r\beta} + iC_{\alpha\alpha}C_{\alpha\beta} - C_{\alpha\alpha}C_{\alpha\beta}$$

$$= C_{\alpha\beta}(C_{\beta\beta} - C_{\alpha\alpha})(1-i)$$

$$= C_{\alpha\beta}(c_{\beta\beta} - c_{\alpha\alpha})(1-i)^2 \quad .$$

Thus $MN \neq NM$ unless $C_{\alpha\beta} = 0$ or $c_{\alpha\alpha} = c_{\beta\beta}$ for $\alpha \neq \beta$. Hence as long as we do not satisfy these conditions we can be sure that $MN \neq NM$ and hence that $T = M + iN$ is not normal. Now the $(\alpha,\beta)^{th}$ element of the Hermitian operator $i(MN - NM)$ is

$$i(1-i)^2 C_{\alpha\beta}(c_{\beta\beta} - c_{\alpha\alpha}) = 2C_{\alpha\beta}(c_{\beta\beta} - c_{\alpha\alpha}) \quad .$$

Hence if $x = \{x_1,x_2,\ldots,x_n\}$, $y = \{y_1,y_2,\ldots,y_n\}$ and $y = i(MN - NM)x$, then

$$y_\alpha = 2 \sum_{\beta=1}^{n} c_{\alpha\beta}(c_{\beta\beta} - c_{\alpha\alpha}) x_\beta = 2 \sum_{\beta=1}^{n} c_{\alpha\beta}(c_{\beta\beta} - c_{\alpha\alpha}) x_\beta \quad .$$

Now let $[\cdot,\cdot]$ be a given semi-inner product on X. Let $x \in X$ where $\|x\| = 1$. Then $[y,x]$ is a linear function of y, say $f(y)$, such that

$$\|f\| = \|x\| = 1 \quad .$$

Thus we can write [see 18, page 37]

$$[y,x] = f(y)$$

$$= \sum_{\alpha=1}^{n} y_\alpha f(\delta^\alpha) \quad ,$$

where δ^α is the unit vector in X with 1 for its α^{th} component.

Hence

$$[i(MN-NM)x,x] = 2 \sum_{\alpha=1}^{n} f(\delta^\alpha) \cdot \sum_{\beta=1}^{n} c_{\alpha\beta}(c_{\beta\beta} - c_{\alpha\alpha}) x_\beta \quad .$$

Let

$$f(\delta^\alpha) = F_\alpha + i\, G_\alpha$$

$$x_\beta = \ell_\beta + i\, m_\beta \quad .$$

Then since $[i(MN-NM)x,x]$ is real, we must have

$$[i(MN-NM)x,x] = 2 \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} c_{\alpha\beta}(c_{\beta\beta}-c_{\alpha\alpha})(F_\alpha \ell_\beta - G_\alpha m_\beta) \quad .$$

Now

$$\begin{aligned} |F_\alpha \ell_\beta - G_\alpha m_\beta| &\leq 2\ |x_\beta| \cdot |f(\delta^\alpha)| \\ &\leq 2\ \|x\| \cdot \|f\| \cdot \|\delta^\alpha\| \\ &= 2 \quad . \end{aligned}$$

Hence

$$-2 \leq F_\alpha \ell_\beta - G_\alpha m_\beta \leq 2 \quad \text{for all } \alpha \text{ and } \beta \ .$$

Let

$$m = \inf_{\alpha,\beta} \ (F_\alpha \ell_\beta - G_\alpha m_\beta) \quad .$$

Then $-2 \leq m \leq 2$.

(i) Suppose $m \geq 0$.

For each β , choose

$$c_{\alpha\beta} = \begin{cases} 0 \ , & \text{if } \alpha < \beta \\ ar^{\alpha+\beta-2} \ , \quad 0 < r < 1, \quad a > 0 \ , & \text{if } \alpha > \beta \ . \end{cases}$$

With this choice of $c_{\alpha\beta}$ we have $MN \neq NM$ and so T is not normal.

Now we have

$$[i(MN-NM)x,x] = 2\sum_{\alpha=1}^{n} \sum_{\beta<\alpha} a^2 r^{\alpha+\beta-2}(r^{2\beta-2}-r^{2\alpha-2})(F_\alpha \ell_\beta - G_\alpha m_\beta) \quad .$$

Since $0 < r < 1$, we have

$$r^{2\beta-2}-r^{2\alpha-2} > 0 \quad \text{for} \quad \beta < \alpha \quad .$$

Thus since $m \leq F_\alpha \ell_\beta - G_\alpha m_\beta$ for all α and β , it follows that

$$[i(MN-NM)x,x] \geq 2\cdot m \sum_{\alpha=1}^{n} \sum_{\beta<\alpha} a^2 r^{\alpha+\beta-2}(r^{2\beta-2}-r^{2\alpha-2})$$

$$\geq 0 \quad .$$

Since x is arbitrary with $\|x\| = 1$, we have thus shown that $T = M + iN$ is not normal and $W\{i(MN-NM)\} \geq 0$

(ii) Suppose $m < 0$.

For each β , choose

$$c_{\alpha\beta} = \begin{cases} 0 \quad , & \text{if } \alpha = \beta \\ 0 \, , & \text{if } \alpha \neq \beta \, , \; \alpha \geq 3, \; \beta \geq 3 \end{cases}$$

$$c_{12} > 0 \, , \quad c_{21} > 0 \, .$$

With this choice we have $MN = NM$ so that

$$W\{i(MN-NM)\} = 0 \quad .$$

We shall show that $T = M + iN$ is not normal because M^2 is not hermitian.

$$M = \begin{Vmatrix} \begin{matrix} ic_{21}, & c_{12}\ , \\ c_{21}, & ic_{12}\ , \end{matrix} & O \\ O & 0 \end{Vmatrix}$$

Hence

$$M^2 = \begin{Vmatrix} \begin{matrix} c_{21}(c_{12}-c_{21})\ , & ic_{12}(c_{21}+c_{12})\ , \\ ic_{21}(c_{21}+c_{12}), & c_{12}(c_{21}-c_{12})\ , \end{matrix} & O \\ O & 0 \end{Vmatrix}$$

Hence

$$\|I+itM^2\| = \max \begin{cases} |1+itc_{21}(c_{12}-c_{21})| + tc_{21}(c_{21}+c_{12}) \, , \\ |1+itc_{12}(c_{21}-c_{12})| + tc_{12}(c_{21}+c_{12}) \, , \\ 1 \, , \end{cases}$$

$$= \max \begin{cases} 1+o(t)+tc_{21}(c_{21}+c_{12}) \, , & \text{as} \quad t \to o \\ 1+o(t)+tc_{12}(c_{21}+c_{12}) \, , & \text{as} \quad t \to o \\ 1+o(t) \, , & \text{as} \quad t \to o \end{cases} .$$

Since $c_{21} > 0$, $c_{12} > 0$,

$$\|I+itM^2\| \neq 1+o(t) \quad \text{as} \quad t \to o \; .$$

Hence M^2 is not hermitian.

Thus by a suitable choice of the $c_{\alpha\beta}$, we can ensure that $T = M + iN$ is not normal and $W\{i(MN-NM)\} \geq 0$.

Now

$$T = M + iN$$

$$= \left\| \begin{matrix} (1+i)C_{11} \, , & & O \\ & (1+i)C_{22} \, , & \\ & O & \end{matrix} \right\|$$

Suppose (S_1, S_2) is a pair of subspaces of X which reduces T completely.

Then $X = S_1 \oplus S_2$.

We may assume without loss of generality that S_1 is spanned by $\delta^1, \delta^2, \ldots, \delta^k$. Then

$$T|_{s_1} = \begin{Vmatrix} (1+i)c_{11}, & & & & \\ & ,\ (1+i)c_{22}\ , & & \bigcirc & \\ & & \ddots & & \\ & \bigcirc & & ,\ (1+i)c_{kk}\ , & \\ & & & & \bigcirc \end{Vmatrix}$$

Thus if $\lambda = 0$ or $\lambda \in \sigma(T)$,

$$\begin{aligned} \|(T-\lambda)|_{s_1}\| &= \max_{1 \leq \alpha \leq k} \{|(1+i)c_{\alpha\alpha} - \lambda|\} \\ &= \max\ [|\lambda| : \lambda \in P\sigma\{(T-\lambda)|_{s_1}\}] \\ &= \max\ [|\lambda| : \lambda \in \sigma\{(T-\lambda)|s_1\}] \\ &= r\{(T-\lambda)|_{s_1}\} \quad . \end{aligned}$$

Similarly we can show that for $\lambda = 0$ or $\lambda \in \sigma(T)$,

$$\|(T-\lambda)|_{s_2}\| = r\{(T-\lambda)|_{s_2}\} \quad .$$

This shows that $T = M + iN$ is hyponormal but not normal.

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